Matrix Versions of Some Refinements of the Arithmetic-Geometric Mean Inequality

Bao Qi Feng and Andrew Tonge

Abstract. We establish matrix versions of refinements due to Alzer [1], Cartwright and Field [4], and Mercer [5] of the standard arithmetic-geometric-harmonic mean inequality for scalars.

1. Introduction

The classical arithmetic-geometric mean inequality asserts that if $w_1, \dots, w_k$ are positive numbers summing to 1 and $x_1, \dots, x_k$ are positive numbers, then the arithmetic mean

$$A_w = A_w(x_1, \dots, x_k) := w_1 x_1 + \dots + w_k x_k$$

is at least as great as the geometric mean

$$G_w = G_w(x_1, \dots, x_k) := x_1^{w_1} \cdots x_k^{w_k}.$$

As this is a consequence of the relatively simple property of convexity of the logarithm function, it is natural to expect more complex and precise relationships to exist between A_w and G_w. Indeed, many authors (for example, Alzer [1], Mercer [5]) have refined the inequality $A_w - G_w \geq 0$. A particularly interesting result, due to Cartwright and Field [4], gives both upper and lower bounds for $A_w - G_w$ in terms of the variance associated with the arithmetic mean.

Theorem 1 (Cartwright-Field) *Let w_i $(1 \leq i \leq k)$ be positive numbers summing to 1. If x_i $(1 \leq i \leq k)$ are positive numbers in the interval $[a, b]$, where $a > 0$, then*

$$\frac{1}{2b} \sum_{j=1}^{k} w_j (x_j - A_w)^2 \leq A_w - G_w \leq \frac{1}{2a} \sum_{j=1}^{k} w_j (x_j - A_w)^2.$$

Cartwright and Field noted that their inequality is sharp, in the sense that there may be equality on both sides.

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Recently, Mercer [5] discovered an extensive collection of inequalities with the same general flavor as those of Cartwright and Field. For now, our emphasis is on extending the Cartwright-Field inequality to positive matrices. It is reasonable to hope that such an extension exists, since Ando [2] and Sagae and Tanabe [7] succeeded in establishing an arithmetic-geometric mean inequality for positive matrices. We recall that the $n \times n$ matrix M is positive if the inner product $< Mx, x >$ is positive for all non-zero complex n-vectors x.

2. The geometric mean of positive matrices

One problem inherent in working with the geometric mean of positive $n \times n$ matrices (at least in the case of non-commuting matrices) is finding an appropriate definition. A major stumbling block is that if $M_1,\ M_2$ are positive $n \times n$ matrices, their product $M_1 M_2$ need not be positive, and fractional powers cannot be defined adequately.

Ando's definition [2] (but see also [6]) of the geometric mean of two equally weighted, (i.e. $w_1 = w_2 = 1/2$) positive matrices $M_1,\ M_2$ was

$$G(M_1, M_2) := M_2^{\frac{1}{2}} \left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}} \right)^{\frac{1}{2}} M_2^{\frac{1}{2}},$$

where all square roots are positive square roots. By design, $G(M_1, M_2) > 0$. Remarkably, in spite of the apparent asymmetry of the definition,

$$G(M_1, M_2) = G(M_2, M_1).$$

This commutativity property is a consequence of an important extremal property [2, 6]: $G(M_1, M_2)$ is the least positive $n \times n$ matrix M with the property that

$$\begin{pmatrix} M_1 & M \\ M & M_2 \end{pmatrix} \geq 0.$$

It is interesting, however, to give a simple direct proof, that does not seen to have been noted before.

Observe that $G(M_1, M_2) = G(M_2, M_1)$ is equivalent to

$$M_1^{-\frac{1}{2}} M_2^{\frac{1}{2}} \left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}} \right)^{\frac{1}{2}} M_2^{\frac{1}{2}} M_1^{-\frac{1}{2}} = \left(M_1^{-\frac{1}{2}} M_2 M_1^{-\frac{1}{2}} \right)^{\frac{1}{2}}.$$

As positive matrices are equal if and only if their squares are equal, this is in turn equivalent to

$$M_1^{-\frac{1}{2}} M_2^{\frac{1}{2}} \left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}} \right)^{\frac{1}{2}} \left[M_2^{\frac{1}{2}} M_1^{-1} M_2^{\frac{1}{2}} \right] \left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}} \right)^{\frac{1}{2}} M_2^{\frac{1}{2}} M_1^{-\frac{1}{2}} = M_1^{-\frac{1}{2}} M_2 M_1^{-\frac{1}{2}}.$$

Since the term in square brackets is just $\left(M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}} \right)^{-1}$, the left hand side of the expression above does indeed reduce to the right hand side.

Building on Ando's success in showing that

$$G(M_1, M_2) \le A(M_1, M_2) := \frac{1}{2}M_1 + \frac{1}{2}M_2,$$

Sagae and Tanabe [7] introduced more general geometric means of an arbitrary number of positive $n \times n$ matrices. For positive numbers $w_1, \ldots, w_k$ summing to 1 and positive $n \times n$ matrices $M_1, \ldots, M_k$, they defined the geometric mean $G_w = G_w(M_1, \ldots, M_k)$ to be

$$M_k^{\frac{1}{2}}(M_k^{-\frac{1}{2}}M_{k-1}^{\frac{1}{2}} \cdots (M_3^{-\frac{1}{2}}M_2^{\frac{1}{2}}(M_2^{-\frac{1}{2}}M_1M_2^{-\frac{1}{2}})^{u_1}M_2^{\frac{1}{2}}M_3^{-\frac{1}{2}})^{u_2} \cdots M_{k-1}^{\frac{1}{2}}M_k^{-\frac{1}{2}})^{u_{k-1}}M_k^{\frac{1}{2}},$$

where $u_i = 1 - \left(w_{i+1} \,/\, \sum_{j=1}^{i+1} w_j \right)$ for $i = 1, \ldots,\ k-1$. All powers are to be interpreted as positive powers, so G_w is easily seen to be positive. If $n = 2$, then

$$G_w(M_1, M_2) = M_2^{\frac{1}{2}}(M_2^{-\frac{1}{2}}M_1M_2^{-\frac{1}{2}})^{w_1}M_2^{\frac{1}{2}},$$

which is consistent with Ando's definition of the geometric mean of two matrices in the case $w_1 = w_2 = 1/2$.

Sagae and Tanabe showed that with the natural definition of the weighted matrix arithmetic mean as

$$A_w(M_1, \ldots, M_k) := w_1M_1 + \cdots + w_kM_k,$$

the matrix analog of the arithmetic-geometric mean inequality is true. In other words,

$$G_w(M_1, \ldots, M_k) \le A_w(M_1, \ldots, M_k).$$

Nevertheless, their matrix geometric mean has a drawback that is potentially problematic in the search for more refined inequalities: the matrix geometric mean of more than two matrices is order dependent. Since this does not appear to have been observed before, we give a simple example generated by *Mathematica*.

We work with the case $k = 3$ and equal weights $w_1 = w_2 = w_3 = 1/3$. Two possible geometric means of positive matrices M_1, M_2 and M_3 are

$$G(M_3, M_2, M_1) = M_1^{1/2}\left(M_1^{-1/2}M_2^{1/2}\left(M_2^{-1/2}M_3M_2^{-1/2}\right)^{1/2}M_2^{1/2}M_1^{-1/2}\right)^{2/3}M_1^{1/2}$$

and

$$G(M_1, M_2, M_3) = M_3^{1/2}\left(M_3^{-1/2}M_2^{1/2}\left(M_2^{-1/2}M_1M_2^{-1/2}\right)^{1/2}M_2^{1/2}M_3^{-1/2}\right)^{2/3}M_3^{1/2}.$$

But, if

$$M_1 := \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, \qquad M_2 := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_3 := \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

then, after computing with *Mathematica*, we find

$$G(M_3,M_2,M_1)=\begin{pmatrix} 1.64446 & 0.614542\\ 0.614542 & 1.19496\end{pmatrix},\quad G(M_1,M_2,M_3)=\begin{pmatrix} 1.61321 & 0.605703\\ 0.605703 & 1.21141\end{pmatrix}.$$

In other words, $G(M_1,M_2,M_3)\neq G(M_3,M_2,M_1)$.

3. Matrix versions of the cartwright-field inequality

Our first objective is to establish a version of the Cartwright-Field inequality for two positive matrices. The proof will depend on a result equivalent to special case of the original inequality with $x_1=t,\ x_2=1$. We give a proof that is different from those published previously.

Lemma 1 *Let $w_1>0,\ w_2>0$ satisfy $w_1+w_2=1$. If $t\in(0,1]$, then*

$$w_1t^2+w_2t-t^{w_1+1}\leq\frac{1}{2}\,w_1w_2(t-1)^2\ \leq w_1t+w_2-t^{w_1}.$$

Equality holds if and only if $t=1$.

Proof We first prove the right hand inequality. Consider

$$f(t)=t^{w_1}-w_1t-w_2+\frac{1}{2}\,w_1w_2(t-1)^2.$$

Since

$$f''(t)=w_1(w_1-1)t^{w_1-2}+w_1w_2=w_1w_2(1-t^{w_1-2})\leq 0$$

for all $t\in(0,1]$, the first derivative f' is decreasing on $(0,1]$, and so $f'(t)\geq f'(1)=0$ for all $t\in(0,1]$. This implies that f is increasing on $(0,1]$, and so $f(t)\leq f(1)=0$ for all $t\in(0,1]$. We have thus established the right hand inequality. It is obvious that equality holds if and only if $t=1$.

The left hand inequality is proved similarly. Consider

$$g(t)=t^{w_1+1}-w_1t^2-w_2t+\frac{1}{2}\,w_1w_2(t-1)^2,$$

then

$$g'(t)=(w_1+1)t^{w_1}-2w_1t-w_2+w_1w_2(t-1),$$
$$g''(t)=w_1(w_1+1)t^{w_1-1}-2w_1+w_1w_2,$$

and hence

$$g'''(t)=(w_1+1)w_1(w_1-1)t^{w_1-2}<0$$

for all $t\in(0,1]$, the second derivative g'' is decreasing on $(0,1]$, and so $g''(t)\geq g''(1)=0$ for all $t\in(0,1]$. This implies that g' is increasing on $(0,1]$, and so $g'(t)\leq g'(1)=0$ for all $t \in (0,1]$. Hence, g is decreasing on $(0,1]$, and $g(t) \geq g(1) = 0$ for all $t \in (0,1]$. We have therefore proved the left hand inequality. Again it is obvious that equality holds if and only if $t = 1$. $\square$

With the aid of this lemma, we can refine a method of Ando [2] to prove a matrix version of the Cartwright-Field inequality for two positive matrices.

Theorem 2 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$. Let M_1, M_2 be positive $n \times n$ matrices with $M_1 \leq M_2$. Write $A_w = A_w(M_1, M_2)$ and $G_w = G_w(M_1, M_2)$. Then*

$$\frac{1}{2}\sum_{j=1}^{2} w_j(M_j - A_w)M_2^{-1}(M_j - A_w) \leq A_w - G_w \leq \frac{1}{2}\sum_{j=1}^{2} w_j(M_j - A_w)M_1^{-1}(M_j - A_w).$$

Equality holds if and only if $M_1 = M_2$.

Proof We start with the left hand inequality. The fact that $M_1 - A_w$ and $M_2 - A_w$ are proportional will allow us to work with a 'single variable' $N := M_2^{-\frac{1}{2}} M_1 M_2^{-\frac{1}{2}}$. Observe that

$$\begin{aligned}
& \sum_{j=1}^{2} w_j(M_j - A_w)M_2^{-1}(M_j - A_w) \\
= \;& w_1(w_2 M_1 - w_2 M_2)M_2^{-1}(w_2 M_1 - w_2 M_2) + w_2(w_1 M_2 - w_1 M_1)M_2^{-1}(w_1 M_2 - w_1 M_1) \\
= \;& w_1 w_2 (w_1 + w_2)(M_1 - M_2)M_2^{-1}(M_1 - M_2) \\
= \;& w_1 w_2 M_2^{\frac{1}{2}}(I - N)^2 M_2^{\frac{1}{2}},
\end{aligned}$$

where I is the $n \times n$ identity matrix. Note that

$$A_w = M_2^{\frac{1}{2}}(w_1 N + w_2 I)M_2^{\frac{1}{2}}$$

and

$$G_w = M_2^{\frac{1}{2}} N^{w_1} M_2^{\frac{1}{2}}.$$

Thus, to prove that

$$\frac{1}{2}\sum_{j=1}^{2} w_j(M_j - A_w)M_2^{-1}(M_j - A_w) \leq A_w - G_w,$$

it is enough to establish

$$\frac{w_1 w_2}{2} M_2^{\frac{1}{2}}(I - N)^2 M_2^{\frac{1}{2}} \leq M_2^{\frac{1}{2}}(w_1 N + w_2 I)M_2^{\frac{1}{2}} - M_2^{\frac{1}{2}} N^{w_1} M_2^{\frac{1}{2}}.$$

This is true if and only if

$$\frac{w_1 w_2}{2}(I-N)^2 \leq w_1 N + w_2 I - N^{w_1}.$$

Note that, since $M_1 \leq M_2$, N is a positive matrix with $N \leq I$. Now $N = U^* D U$, where D is a diagonal matrix $[d_1, \cdots, d_n]$ and U is a unitary matrix. Since $0 < N \leq I$, it follows that $0 < D \leq I$ and so $0 < d_i \leq 1$ $(1 \leq i \leq n)$. By Lemma 1, we have

$$\frac{w_1 w_2}{2}(1-d_i)^2 \leq w_1 d_i + w_2 - d_i^{w_1} \qquad (1 \leq i \leq n).$$

So

$$\frac{w_1 w_2}{2}(I-D)^2 \leq w_1 D + w_2 I - D^{w_1},$$

and this implies that

$$\frac{w_1 w_2}{2}(I-N)^2 \leq w_1 N + w_2 I - N^{w_1}.$$

Hence

$$\frac{1}{2}\sum_{j=1}^{2} w_j (M_j - A_w) M_2^{-1} (M_j - A_w) \leq A_w - G_w.$$

Equality holds if and only if $D = I$, that is, $N = I$ and $M_1 = M_2$.

Next, we prove the right hand inequality, using similar techniques. A computation shows that

$$\sum_{j=1}^{2} w_j (M_j - A_w) M_1^{-1} (M_j - A_w) = w_1 w_2 M_2^{\frac{1}{2}} (I-N)^2 N^{-1} M_2^{\frac{1}{2}}.$$

Hence, to establish

$$A_w - G_w \leq \frac{1}{2}\sum_{j=1}^{2} w_j (M_j - A_w) M_1^{-1} (M_j - A_w),$$

we need to show

$$M_2^{\frac{1}{2}}(w_1 N + w_2 I) M_2^{\frac{1}{2}} - M_2^{\frac{1}{2}} N^{w_1} M_2^{\frac{1}{2}} \leq \frac{1}{2}\, w_1 w_2 M_2^{\frac{1}{2}} (I-N)^2 N^{-1} M_2^{\frac{1}{2}}.$$

But this is true if and only if

$$w_1 N^2 + w_2 N - N^{w_1+1} \leq \frac{1}{2}\, w_1 w_2 (I-N)^2,$$

and this follows from Lemma 1 via the diagonalization technique above. Again, equality holds if and only if $N = I$, that is $M_1 = M_2$. □

It is natural to ask whether our matrix extension of the Cartwright-Field inequality can be carried over to the case of three or more matrices. If the matrices commute, this is no problem. Indeed, no ordering hypothesis is necessary in the presence of commutativity. Commuting positive matrices can be viewed as elements of a commutative C^* algebra, which, by Gelfand's theorem [3], is isometrically $*$-isomorphic to the algebra $C(K)$ of continuous functions on an appropriate compact Hausdorff space K. The positive matrices then correspond to positive functions, and the scalar inequality of Cartwright and Field can be applied at each point of K. When we drop the commutativity hypothesis, we are unable to prove analogs of Theorem 2 for more than two matrices. This is possibly related to general problems with the geometric mean of more than two matrices that we illustrated above.

4. Matrix refinements of the arithmetic-geometric-harmonic mean inequality

Recall that if $w_1, \dots, w_k$ are positive number summing to 1, the weighted harmonic mean of the positive numbers $x_1, \dots, x_k$ is

$$H_w := H_w(x_1, \dots, x_k) = (w_1 x_1^{-1} + \cdots + w_k x_k^{-1})^{-1}.$$

The classical result that

$$H_w \leq G_w \leq A_w$$

has been extensively refined. Building on a result of Alzer [1], Mercer [5] found an extensive collection of inequalities that he proved using a global technique. We summarize some of his results.

Theorem 3 (Mercer) *Let x_i $(1 \leq i \leq k)$ be real numbers, ordered so that $0 < x_1 \leq \cdots \leq x_k$, and let w_i $(1 \leq i \leq k)$ be positive numbers with $\sum_{i=1}^{k} w_i = 1$. Then, if we write $A_w = A_w(x_1, \dots, x_k)$, $G_w = G_w(x_1, \dots, x_k)$, and $H_w = H_w(x_1, \dots, x_k)$, we have*

(1) $\frac{1}{2x_k} \sum_{j=1}^{k} w_j (x_j - G_w)^2 \leq A_w - G_w \leq \frac{1}{2x_1} \sum_{j=1}^{k} w_j (x_j - G_w)^2$;

(2) $\frac{1}{2x_k} \sum_{j=1}^{k} w_j (x_j - H_w)^2 \leq A_w - H_w$;

(3) $\frac{G_w}{2x_k^2} \sum_{j=1}^{k} w_j (x_j - G_w)^2 \leq G_w - H_w \leq \frac{G_w}{2x_1^2} \sum_{j=1}^{k} w_j (x_j - G_w)^2$;

(4) $\frac{G_w}{2x_k^2} \sum_{j=1}^{k} w_j (x_j - H_w)^2 \leq G_w - H_w \leq \frac{G_w}{2x_1^2} \sum_{j=1}^{k} w_j (x_j - H_w)^2$.

These inequalities are strict unless all the x_i are equal.

The right hand side of (1) had been established by Alzer [1].

A natural question is whether a version of Mercer's theorem holds for positive matrices. We are able to provide a positive answer in the case of two matrices?

Lemma 2 *Let* $w_1 > 0,\ w_2 > 0$ *satisfy* $w_1 + w_2 = 1,$ *and* $t \in (0, 1],$ *then*

$$w_1 t^2 + w_2 t - t^{w_1+1} \le \frac{1}{2}\ \left[w_1(t - t^{w_1})^2 + w_2(1 - t^{w_1})^2\right] \le w_1 t + w_2 - t^{w_1}. \qquad (1')$$

Equality holds if and only if $t = 1$.

Proof We start to prove the right hand inequality of $(1')$. First of all, we have

$$\begin{aligned} &\frac{1}{2}\ \left[w_1(t - t^{w_1})^2 + w_2(1 - t^{w_1})^2\right] \le w_1 t + w_2 - t^{w_1} \\ \Longleftrightarrow\quad & -\frac{1}{2}\ w_2 - w_1 t + \frac{1}{2}\ w_1 t^2 + w_1 t^{w_1} - w_1 t^{w_1+1} + \frac{1}{2}\ t^{2w_1} \le 0. \end{aligned}$$

Let

$$f(t) = -\frac{1}{2}\ w_2 - w_1 t + \frac{1}{2}\ w_1 t^2 + w_1 t^{w_1} - w_1 t^{w_1+1} + \frac{1}{2}\ t^{2w_1}.$$

Then $f(1) = 0$ and

$$f'(t) = -w_1 + w_1 t + w_1^2 t^{w_1-1} - w_1(w_1 + 1)t^{w_1} + w_1 t^{2w_1-1}.$$

Doing more we have that $f'(1) = 0$,

$$f''(t) = w_1 + w_1^2(w_1 - 1)t^{w_1-2} - w_1^2(w_1 + 1)t^{w_1-1} + w_1(2w_1 - 1)t^{2w_1-2},$$

and $f''(1) = 0$. Taking the third derivative of $f(t)$ with simplification, we have

$$f'''(t) = w_1 w_2 t^{w_1-3}[w_1^2 + w_1 w_2 t + 2w_2 t^{w_1} + 2w_1(w_1 t + w_2 - t^{w_1})].$$

Notice that the right hand inequality of Lemma 1 implies

$$w_1 t + w_2 - t^{w_1} > 0,$$

so we have $f'''(t) > 0$, which implies that $f''(t)$ is increasing on $(0, 1]$ and $f''(t) \le f''(1) = 0$, that means $f'(t)$ is decreasing on $(0, 1]$ and $f'(t) \ge f'(1) = 0$. Hence $f(t)$ is increasing on $(0, 1]$ and $f(t) \le f(1) = 0$, which means the right hand inequality of $(1')$ holds.

Similarly, using the same way we prove the left hand inequality of $(1')$. It is easy to check that

$$\begin{aligned} &w_1 t^2 + w_2 t - t^{w_1+1} \le \frac{1}{2}\ \left[w_1(t - t^{w_1})^2 + w_2(1 - t^{w_1})^2\right] \\ \Longleftrightarrow\quad & -\frac{1}{2}\ w_2 + w_2 t + \frac{1}{2}\ w_1 t^2 + w_2 t^{w_1} - w_2 t^{w_1+1} - \frac{1}{2}\ t^{2w_1} \le 0. \end{aligned}$$

Let

$$g(t) = -\frac{1}{2}\ w_2 + w_2 t + \frac{1}{2}\ w_1 t^2 + w_2 t^{w_1} - w_2 t^{w_1+1} - \frac{1}{2}\ t^{2w_1},$$

then $g(1)=0$ and

$$g'(t)=w_2+w_1t+w_1w_2t^{w_1-1}-w_2(w_1+1)t^{w_1}-w_1t^{2w_1-1}.$$

Doing more we have $g'(1)=0$,

$$g''(t)=w_1-w_1w_2^2t^{w_1-2}-w_1w_2(w_1+1)t^{w_1-1}-w_1(2w_1-1)t^{2w_1-2},$$

and $g''(1)=0$. Continuously taking the third derivative of $g(t)$, we have

$$g'''(t)=w_1w_2t^{w_1-3}[w_1w_2+w_2^2t+2w_1t^{w_1}+2w_2(w_1t+w_2-t^{w_1})].$$

By the right hand inequality of Lemma 1, we have $g'''(t)>0$ on $(0,1]$. Hence the left hand inequality of $(1')$ holds. □

The generalization of inequality (1) in Theorem 3 to the case of two matrices is given as follows.

Theorem 4 *Let $w_1>0$, $w_2>0$ satisfy $w_1+w_2=1$. Let M_1, M_2 be positive $n\times n$ matrices with $M_2\geq M_1$. Write $A_w=A_w(M_1,M_2)$ and $G_w=G_w(M_1,M_2)$. Then*

$$\frac{1}{2}\sum_{j=1}^{2}w_j(M_j-G_w)M_2^{-1}(M_j-G_w)\leq A_w-G_w\leq\frac{1}{2}\sum_{j=1}^{2}w_j(M_j-G_w)M_1^{-1}(M_j-G_w).$$

Equality holds if and only if $M_1=M_2$.

Proof We start to prove the left side of the inequality. First of all, we have

$$M_1-G_w=M_2^{\frac{1}{2}}(N-N^{w_1})M_2^{\frac{1}{2}}\quad and\quad M_2-G_w=M_2^{\frac{1}{2}}(I-N^{w_1})M_2^{\frac{1}{2}},$$

which imply that

$$\begin{aligned}&\sum_{j=1}^{2}w_j(M_j-G_w)M_2^{-1}(M_j-G_w)\\ =\;&w_1M_2^{\frac{1}{2}}(N-N^{w_1})^2M_2^{\frac{1}{2}}+w_2M_2^{\frac{1}{2}}(I-N^{w_1})^2M_2^{\frac{1}{2}}.\\ =\;&M_2^{\frac{1}{2}}\left[w_1(N-N^{w_1})^2+w_2(I-N^{w_1})^2\right]M_2^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{2}w_j(M_j-G_w)M_2^{-1}(M_j-G_w)\leq A_w-G_w\\ \Longleftrightarrow\;&\frac{1}{2}M_2^{\frac{1}{2}}\left[w_1(N-N^{w_1})^2+w_2(I-N^{w_1})^2\right]M_2^{\frac{1}{2}}\leq M_2^{\frac{1}{2}}(w_1N+w_2I-N^{w_1})M_2^{\frac{1}{2}}\\ \Longleftrightarrow\;&\frac{1}{2}\left[w_1(N-N^{w_1})^2+w_2(I-N^{w_1})^2\right]\leq w_1N+w_2I-N^{w_1},\end{aligned}$$

which holds since the right hand inequality of $(1')$. Equality holds if and only if $N = I$, that is $M_1 = M_2$.

Next, we going to prove the right side of the inequality. We have

$$\begin{aligned}
&\sum_{j=1}^{2} w_j (M_j - G_w) M_1^{-1} (M_j - G_w) \\
= \;& w_1 M_2^{\frac{1}{2}} (N - N^{w_1}) N^{-1} (N - N^{w_1}) M_2^{\frac{1}{2}} + w_2 M_2^{\frac{1}{2}} (I - N^{w_1}) N^{-1} (I - N^{w_1}) M_2^{\frac{1}{2}} \\
= \;& M_2^{\frac{1}{2}} \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] N^{-1} M_2^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
& A_w - G_w \le \frac{1}{2} \sum_{j=1}^{2} w_j (M_j - G_w) M_1^{-1} (M_j - G_w) \\
\iff \;& w_1 N + w_2 I - N^{w_1} \le \frac{1}{2} \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] N^{-1} \\
\iff \;& w_1 N^2 + w_2 N - N^{w_1 + 1} \le \frac{1}{2} \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right],
\end{aligned}$$

which holds since the left hand inequality of $(1')$. Equality holds if and only if $N = I$, that is $M_1 = M_2$. $\square$

For the generalization of inequality (2) in Theorem 3, we need the following lemma.

Lemma 3 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$, and $t \in (0, 1]$, then*

$$\frac{1}{2} \left[w_1 \left(t - \frac{t}{w_1 + w_2 t} \right)^2 + w_2 \left(1 - \frac{t}{w_1 + w_2 t} \right)^2 \right] \le w_1 t + w_2 - \frac{t}{w_1 + w_2 t}. \qquad (2')$$

Equality holds if and only if $t = 1$.

Proof After algebraic simplification, we have

$$w_1 \left(t - \frac{t}{w_1 + w_2 t} \right)^2 + w_2 \left(1 - \frac{t}{w_1 + w_2 t} \right)^2 = \frac{w_1 w_2 (t-1)^2}{(w_1 + w_2 t)^2} (w_2 t^2 + w_1).$$

Hence

$$\begin{aligned}
&\frac{1}{2}\left[w_1\left(t-\frac{t}{w_1+w_2t}\right)^2+w_2\left(1-\frac{t}{w_1+w_2t}\right)^2\right]\le w_1t+w_2-\frac{t}{w_1+w_2t}\\
\Longleftrightarrow\quad &\frac{w_1w_2(t-1)^2}{2(w_1+w_2t)^2}(w_2t^2+w_1)\le\frac{w_1w_2(1-t)^2}{w_1+w_2t}\\
\Longleftrightarrow\quad &\frac{w_2t^2+w_1}{2(w_1+w_2t)}\le 1\\
\Longleftrightarrow\quad &w_2(t-1)^2\le 1,
\end{aligned}$$

which is obvious. □

The matrix version of inequality (2) in Theorem 3 to the case of two matrices as follows.

Theorem 5 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$. Let M_1, M_2 be positive $n \times n$ matrices with $M_2 \geq M_1$. Write $A_w = A_w(M_1, M_2)$ and $H_w = H_w(M_1, M_2)$. Then*

$$\frac{1}{2}\sum_{j=1}^{2} w_j(M_j-H_w)M_2^{-1}(M_j-H_w)\le A_w-H_w.$$

Equality holds if and only if $M_1 = M_2$.

Proof As before, it is convenient to use $N = M_2^{-\frac{1}{2}}M_1M_2^{-\frac{1}{2}}$. In addition, we set $C = (w_1I + w_2N)^{-1}$. We have already seen that $0 < N \leq I$, when $M_1 \leq M_2$, and it is clear that N and C commute. By the definition of H_w, we have

$$\begin{aligned}
H_w &= \left(w_1M_2^{-\frac{1}{2}}N^{-1}M_2^{-\frac{1}{2}}+w_2M_2^{-\frac{1}{2}}IM_2^{-\frac{1}{2}}\right)^{-1}\\
&= M_2^{\frac{1}{2}}\left(N(w_1I+w_2N)^{-1}\right)M_2^{\frac{1}{2}}\\
&= M_2^{\frac{1}{2}}NCM_2^{\frac{1}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
M_1-H_w &= M_2^{\frac{1}{2}}(N-NC)M_2^{\frac{1}{2}},\\
M_2-H_w &= M_2^{\frac{1}{2}}(I-NC)M_2^{\frac{1}{2}}.
\end{aligned}$$

Thus,

$$\frac{1}{2}\sum_{j=1}^{2} w_j(M_j-H_w)M_2^{-1}(M_j-H_w) \le A_w - H_w$$
$$\iff \frac{1}{2}\left[w_1(N-NC)^2+w_2(I-NC)^2\right] \le w_1N+w_2I-NC,$$

which holds since inequality $(2')$. Equality holds if and only if $N=I$, that is $M_1=M_2$. □

We give the following lemma for generalizing the inequality (3) in Theorem 3.

Lemma 4 *Let $w_1>0$, $w_2>0$ satisfy $w_1+w_2=1$, and $t\in(0,1]$, then*

$$t^2-\frac{t^{w_2+2}}{w_1+w_2t} \le \frac{1}{2}\left[w_1(t-t^{w_1})^2+w_2(1-t^{w_1})^2\right] \le 1-\frac{t^{w_2}}{w_1+w_2t}. \qquad (3')$$

Equality holds if and only if $t=1$.

Proof It is difficult that giving immediately proof for inequalities $(3')$ with the same way of Lemma 1. With the help of inequalities (iii) in Theorem 3, we give the following proof of inequalities $(3')$.

Let $k=2$ in the inequalities (iii) of Theorem 3, we have

$$\frac{G_w}{2x_2^2}\sum_{j=1}^{2} w_j(x_j-G_w)^2 \le G_w-H_w \le \frac{G_w}{2x_1^2}\sum_{j=1}^{2} w_j(x_j-G_w)^2,$$

which is equivalent to

$$\frac{x_1^{w_1}x_2^{w_2}}{2x_2^2}\sum_{j=1}^{2} w_j(x_j-x_1^{w_1}x_2^{w_2})^2 \le x_1^{w_1}x_2^{w_2}-(w_1x_1^{-1}+w_2x_2^{-1})^{-1} \le \frac{x_1^{w_1}x_2^{w_2}}{2x_1^2}\sum_{j=1}^{2} w_j(x_j-x_1^{w_1}x_2^{w_2})^2.$$

Since $x_1\le x_2$, let $t=x_1/x_2$, then $0<t\le 1$, and after a few steps of verifying we have

$$x_1^{w_1}x_2^{w_2}-(w_1x_1^{-1}+w_2x_2^{-1})^{-1} = x_2t^{w_1}\left[1-\frac{t^{w_2}}{w_1+w_2t}\right],$$

$$\frac{x_1^{w_1}x_2^{w_2}}{2x_2^2}\sum_{j=1}^{2} w_j(x_j-x_1^{w_1}x_2^{w_2})^2 = \frac{1}{2}x_2t^{w_1}\left[w_1(t-t^{w_1})^2+w_2(1-t^{w_1})^2\right],$$

and

$$\frac{x_1^{w_1}x_2^{w_2}}{2x_1^2}\sum_{j=1}^{2} w_j(x_j-x_1^{w_1}x_2^{w_2})^2 = \frac{1}{2}x_2t^{w_1-2}\left[w_1(t-t^{w_1})^2+w_2(1-t^{w_1})^2\right].$$

From these we imply that the inequalities $(3')$ holds. □

The matrix version corresponding the inequality (3) in Theorem 3 is the following.

Theorem 6 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$. Let M_1, M_2 be positive $n \times n$ matrices with $M_2 \geq M_1$. Write $G_w = G_w(M_1, M_2)$ and $H_w = H_w(M_1, M_2)$. Then*

$$\frac{1}{2}\, G_w M_2^{-1} \sum_{j=1}^{2} w_j (M_j - G_w) M_2^{-1} (M_j - G_w) \leq G_w - H_w$$

$$\leq \frac{1}{2}\, G_w M_1^{-1} \sum_{j=1}^{2} w_j (M_j - G_w) M_1^{-1} (M_j - G_w).$$

Equality holds if and only if $M_1 = M_2$.

Proof First we prove the left side of the inequality. Using the notation in Theorem 5 we used, we have

$$\begin{aligned}
& \frac{1}{2}\, G_w M_2^{-1} \sum_{j=1}^{2} w_j (M_j - G_w) M_2^{-1} (M_j - G_w) \leq G_w - H_w \\
\iff\ & \frac{1}{2}\, M_2^{\frac{1}{2}} N^{w_1} \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] M_2^{\frac{1}{2}} \leq M_2^{\frac{1}{2}} (N^{w_1} - NC) M_2^{\frac{1}{2}} \\
\iff\ & \frac{1}{2}\, N^{w_1} \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] \leq N^{w_1} - NC \\
\iff\ & \frac{1}{2}\, \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] \leq I - N^{w_2} C,
\end{aligned}$$

which holds since the right hand inequality of $(3')$. Equality holds if and only if $N = I$, that is $M_1 = M_2$.

Now we prove the right side of the inequality. We have

$$\begin{aligned}
& G_w - H_w \leq \frac{1}{2}\, G_w M_1^{-1} \sum_{j=1}^{2} w_j (M_j - G_w) M_1^{-1} (M_j - G_w) \\
\iff\ & N^{w_1} - NC \leq \frac{1}{2}\, \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right] N^{w_1 - 2} \\
\iff\ & N^2 - N^{w_2 + 2} C \leq \frac{1}{2}\, \left[w_1 (N - N^{w_1})^2 + w_2 (I - N^{w_1})^2 \right],
\end{aligned}$$

which holds since the left hand inequality of $(3')$. Equality holds if and only if $N = I$, that is $M_1 = M_2$. $\square$

We turn to the last inequality (4) in Theorem 3. Similarly we give the following lemma.

Lemma 5 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$, and $t \in (0, 1]$, then*

$$t^2 - \frac{t^{w_2+2}}{w_1 + w_2 t} \leq \frac{1}{2} \left[w_1 \left(t - \frac{t}{w_1 + w_2 t} \right)^2 + w_2 \left(1 - \frac{t}{w_1 + w_2 t} \right)^2 \right] \leq 1 - \frac{t^{w_2}}{w_1 + w_2 t}. \quad (4')$$

Equality holds if and only if $t = 1$.

Proof First we prove the right hand inequality of (4′). From the proof of Lemma 3 we have

$$\begin{aligned}
& \frac{1}{2}\left[w_1\left(t-\frac{t}{w_1+w_2t}\right)^2+w_2\left(1-\frac{t}{w_1+w_2t}\right)^2\right]\le 1-\frac{t^{w_2}}{w_1+w_2t}\\
\Longleftrightarrow\quad & \frac{w_1w_2(t-1)^2}{2(w_1+w_2t)^2}(w_2t^2+w_1)\le\frac{(w_1+w_2t)^2-(w_1+w_2t)t^{w_2}}{(w_1+w_2t)^2}\\
\Longleftrightarrow\quad & \frac{1}{2}w_1w_2(1-t)^2(w_2t^2+w_1)\le(w_1+w_2t)(w_1+w_2t-t^{w_2}).
\end{aligned}$$

If $t=1$, this is certainly true, and in fact becomes an equality. If $t\neq 1$, by the right hand inequality of Lemma 1, we have

$$0<\frac{1}{2}w_1w_2(1-t)^2\le w_1+w_2t-t^{w_2}.$$

Consequently, we can achieve our goal by showing that

$$w_1+w_2t^2<w_1+w_2t,$$

when $t\neq 1$. However this follows the facts that $t\in(0,1)$ and $w_2>0$.

Next we prove the left hand inequality of (4′). We know that

$$\begin{aligned}
& t^2-\frac{t^{w_2+2}}{w_1+w_2t}\le\frac{1}{2}\left[w_1\left(t-\frac{t}{w_1+w_2t}\right)^2+w_2\left(1-\frac{t}{w_1+w_2t}\right)^2\right]\\
\Longleftrightarrow\quad & \frac{t^2(w_1+w_2t)^2-t^{w_2+2}(w_1+w_2t)}{(w_1+w_2t)^2}\le\frac{w_1w_2(t-1)^2}{2(w_1+w_2t)^2}(w_2t^2+w_1)\\
\Longleftrightarrow\quad & (w_1t+w_2t^2)(w_1t+w_2t^2-t^{w_2+1})\le\frac{1}{2}w_1w_2(1-t)^2(w_2t^2+w_1).
\end{aligned}$$

If $t=1$, this is certainly true, and in fact becomes an equality. If $t\neq 1$, by the left hand inequality of Lemma 1, we have

$$0<w_1t+w_2t^2-t^{w_2+1}\le\frac{1}{2}w_1w_2(1-t)^2.$$

Since $0<t<1$,

$$w_1t+w_2t^2<w_1+w_2t^2$$

holds, the proof is completed. □

Here is the matrix version of inequality (4) in Theorem 3 to the case of two matrices.

Theorem 7 *Let $w_1 > 0$, $w_2 > 0$ satisfy $w_1 + w_2 = 1$. Let M_1, M_2 be positive $n \times n$ matrices with $M_2 \geq M_1$. Write $G_w = G_w(M_1, M_2)$ and $H_w = H_w(M_1, M_2)$. Then*

$$\frac{1}{2}\, G_w M_2^{-1} \sum_{j=1}^{2} w_1(M_j - H_w) M_2^{-1}(M_j - H_w) \leq G_w - H_w$$

$$\leq \frac{1}{2}\, G_w M_1^{-1} \sum_{j=1}^{2} w_1(M_j - H_w) M_1^{-1}(M_j - H_w).$$

Equality holds if and only if $M_1 = M_2$.

Proof It is similar with Theorem 5, we have

$$\begin{aligned}
& \frac{1}{2}\, G_w M_2^{-1} \sum_{j=1}^{2} w_1(M_j - H_w) M_2^{-1}(M_j - H_w) \leq G_w - H_w \\
\iff\ & \frac{1}{2}\, N^{w_1} \left[w_1(N - NC)^2 + w_2(I - NC)^2\right] \leq N^{w_1} - NC \\
\iff\ & \frac{1}{2}\, \left[w_1(N - NC)^2 + w_2(I - NC)^2\right] \leq I - N^{w_2}C,
\end{aligned}$$

which holds since the right hand inequality of (4′). Equality holds if and only if $N = I$, that is $M_1 = M_2$.

For the right hand inequality, we have

$$\begin{aligned}
& G_w - H_w \leq \frac{1}{2}\, G_w M_1^{-1} \sum_{j=1}^{2} w_1(M_j - H_w) M_1^{-1}(M_j - H_w) \\
\iff\ & N^{w_1} - NC \leq \frac{1}{2}\, N^{w_1 - 2} \left[w_1(N - NC)^2 + w_2(I - NC)^2\right] \\
\iff\ & N^2 - N^{w_2 + 2}C \leq \frac{1}{2}\, \left[w_1(N - NC)^2 + w_2(I - NC)^2\right],
\end{aligned}$$

which holds since the left hand inequality of (4′). Equality holds if and only if $N = I$, that is $M_1 = M_2$. □

As with the Cartwright-Field inequality, Theorem 4 to Theorem 7 can be extended (without the need for an ordering hypothesis) to the case of several commuting positive $n \times n$ matrices. It would be interesting to know whether the commutativity or ordering hypothesis can be dropped.

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Bao Qi Feng
Department of Mathematical Sciences
Kent State University, Tuscarawas Campus
New Philadelphia, OH 44663-9403, USA
e-mail: bfeng@kent.edu

Andrew Tonge
Department of Mathematical Sciences
Kent State University
Kent, OH 44240, USA
email: tonge@math.kent.edu